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Euler Characteristic and Related Measures for Random Geometric Sets

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By an elementary calculation we obtain the exact mean values of Minkowksi functionals for a standard model of percolating sets. In particular, a recurrence theorem for the mean Euler characteristic recently put forward is shown to be incorrect. Related previous mathematical work is mentioned. We also conjecture bounds for the threshold density of continuum percolation, which are associated with the Euler characteristic.

KEY WORDS: Random sets; integral geometry; Minkowski functionals; Euler characteristic; continuum percolation.

In a recent communication,[1] Okun stressed the role of the Euler characteristic in percolation theory. For the canonical model of percolation in the continuum, where penetrable balls are randomly placed in $\mathbb{R}^d$, he obtained explicit and simple expressions for the mean Euler characteristic χ in arbitrary dimensions $d \geqslant 1$. This result is interesting since χ is an important topological descriptor for random geometric sets which is employed in such diverse fields as, for instance, image analysis,[2] microemulsions,[3,4] and the large-scale distribution of galaxies.[5]

Stimulated by Okun's work, we considered a more general model, where the balls are replaced by penetrable grains with random size and shape. We computed the exact mean values for a family of $d+1$ morphological measures, including the Euler characteristic as a prominent member. These additive and motion invariant measures are known in integral geometry[6,8] as Quermass integrals or Minkowski functionals. We also found that Okun's recurrence formula[1] for χ is erroneous.

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While preparing our manuscript, we became aware of the mathematical literature dealing with similar topics in the context of stochastic geometry and stereology (see, e.g., ref. 7). In particular, our results have previously been obtained by Davy[(9)] and, within a more abstract setting, by Kellerer.[(10)] Since this issue might not be so familiar and in order to correct the error in ref. 1, we present here our elementary computation of the mean Minkowski functionals for the graim model.

We also observe by comparison with numerical data that, at least for certain grain shapes, the mean Euler characteristic vanishes at a density close to the threshold of continuum percolation.

Let us first recall some basic facts from combinatorial integral geometry.[2] The convex ring $\mathscr{R}$ constitutes the stage for our model. $\mathscr{R}$ denotes the class of all subsets A of the Euclidean space $\mathbb{R}^d$ which can be represented in terms of a finite union of bounded closed convex sets; also, $\varnothing \in \mathscr{R}$. The Euler characteristic χ is introduced as an additive functional over $\mathscr{R}$, so that for $A, B \in \mathscr{R}$,

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B) \tag{1}$$

and

$$\chi(A) = \begin{cases} 1, & \text{convex } A \neq \varnothing \\ 0, & A = \varnothing \end{cases} \tag{2}$$

We note that this functional χ coincides with the Euler characteristic in algebraic topology, employed in ref. 1.

The Minkowski functionals over $\mathscr{R}$ are now defined through

$$\begin{aligned} W_\alpha(A) &= \int \chi(A \cap E_\alpha)\, d\mu(E_\alpha), & \alpha &= 0,\dots, d-1 \\ W_d(A) &= \omega_d \chi(A), & \omega_d &= \pi^{d/2}/\Gamma(1+d/2) \end{aligned} \tag{3}$$

Here, E_α is an α-dimensional plane in $\mathbb{R}^d$, $d\mu(E_\alpha)$ denotes its kinematical density normalized so that for a d-dimensional ball $B_d(r)$ with radius r, $W_\alpha(B_d(r)) = \omega_d r^{d-\alpha}$; ω_d is the volume of the unit ball.

From definition (3) it is clear that the Minkowski functionals inherit additivity from χ. It will be convenient to renormalize by setting

$$M_\alpha(A) = \frac{\omega_{d-\alpha}}{\omega_\alpha \omega_d} W_\alpha(A) \tag{4}$$

In Table I, the functionals M_α for $d \leqslant 3$ are expressed in more familiar terms.

[2] We follow the exposition in Chapter 6 of ref. 6.

Table I. Renormalized Minkowski Functionals for d-Dimensional Sets $A^d \in \mathscr{R}$ in Terms of Geometric Measures[a]

d	M_0	M_1	M_2	M_3
1	L	$\chi_1/2$	—	—
2	F	$U/2\pi$	χ_2/π	—
3	V	$S/8$	$H/2\pi^2$	$3\chi_3/4\pi$

[a] L, length; F, area; U, circumference; V, volume; S, surface area; H, integral mean curvature; $\chi_d = \chi(A^d)$.

Let $\mathscr{G}$ be the group of motions (translations and rotations) in the Euclidean space $\mathbb{R}^d$. The action of $g \in \mathscr{G}$ on the set A is denoted by gA. Since χ and the kinematical densities are invariant under $\mathscr{G}$, we also have $M_\alpha(gA) = M_\alpha(A)$, $\alpha = 0,..., d$.

A remarkable result of integral geometry is Hadwiger's theorem, which states that the family of $d+1$ Minkowski functionals is complete in the following sense: If F is an additive, motion-invariant, and continuous (or monotonous) functional over $\mathscr{R}$, then $F = \sum_{\alpha=0}^{d} f_\alpha M_\alpha$ with suitable coefficients $\{f_\alpha\}$. Hadwiger's theorem is an essential element in his proof of the "principal kinematical formulas"

$$\int_{\mathscr{G}} M_\alpha(A \cap gB)\, dg = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} M_{\alpha-\beta}(B)\, M_\beta(A), \qquad \alpha = 0,..., d \tag{5}$$

In modern terminology, the kinematical density[3] dg in (5) is the Haar measure of the group $\mathscr{G}$.

In terms of the Minkowski polynomials introduced in ref. 10,

$$M(A) = \sum_{\alpha=0}^{d} \frac{t^\alpha}{\alpha!} M_\alpha(A) \tag{6}$$

Eqs. (5) can be summarized in the concise form

$$\int_{\mathscr{G}} M(A \cap gB)\, dg \sim M(A)\, M(B) \tag{7}$$

where the tilde denotes equality up to $O(t^d)$.

In passing we mention[(11)] that there is a natural way to formulate a discrete integral geometry for polyhedral sets on regular lattices; we shall

[3] In Hadwiger's notation,[(6)] $dg = dB/c_d$, $c_d = (d!/2) \prod_{k=1}^{d} \omega_k$.

not pursue these matters here, but turn now to the main topic of the present note.

We draw grains $A_i \in \mathscr{R}$, $i = 1, \ldots, N$, from an ensemble with finite values for all $M_\alpha(A_i)$. At the centroid of each grain we fix a d-Bein. These so-marked penetrable grains are placed independently at N random sites in a d-cube Ω with random isotropic orientation of their d-Beins. To avoid edge effects in applying (7), we use periodic boundary conditions on $\partial\Omega$. Thus, a random configuration of grains gives rise to a set

$$\mathscr{A}_N = \bigcup_{i=0}^{N} g_i A_i \in \mathscr{R} \tag{8}$$

Our aim is to compute the mean value of $M(\mathscr{A}_N \cap D)$, where $D \subset \Omega$ is a convex test domain. The configurational average is done with the product density element

$$d\mu(g_1, \ldots, g_N) = \frac{1}{|\Omega|^N} \prod_{i=1}^{N} dg_i \tag{9}$$

$\int dg_i = |\Omega| = \mathrm{vol}(\Omega)$, where the integration over translations is restricted to Ω.

Consider first the configurational average for a single grain, A_N. Additivity of M combined with the kinematic formula (7) and with $M(g_N A_N) = M(A_N)$ leads to

$$\begin{aligned}
&\int M(\mathscr{A}_N \cap D)\, dg_N/|\Omega| \\
&\qquad = M(\mathscr{A}_{N-1} \cap D) + \int [M(D \cap g_N A_N) \\
&\qquad\quad - M(\mathscr{A}_{N-1} \cap D \cap g_N A_N)]\, dg_N/|\Omega| \\
&\qquad \sim M(\mathscr{A}_{N-1} \cap D) + [M(D) - M(\mathscr{A}_{N-1} \cap D)]\, M(A_N)/|\Omega|
\end{aligned} \tag{10}$$

The further average over size and shape of A_N replaces $M(A_N)$ by its mean value m.

Since the grains are independently and identically distributed in location, orientation, size, and shape, Eq. (10) leads to a simple difference equation (mod t^{d+1}) for the mean value $\bar{M}_N = \bar{M}_N(m, D, \Omega)$ of $M(\mathscr{A}_N \cap D)$,

$$\bar{M}_N - \left(1 - \frac{m}{|\Omega|}\right) \bar{M}_{N-1} \sim M(D) \frac{m}{|\Omega|} \tag{11}$$

which is solved with the initial value $\bar{M}_{N=1} = M(D)m/|\Omega|$ by

$$\bar{M}_N \sim M(D)\left[1 - \left(1 - \frac{m}{|\Omega|}\right)^N\right] \tag{12}$$

In the bulk limit N, $|\Omega| \to \infty$, $\rho = N/|\Omega|$ fixed, we arrive at

$$\bar{M}(\mu, D) \sim M(D)(1 - e^{-\mu}) \tag{13}$$

where $\mu = \rho m = \sum_{\alpha=0}^{d} \rho m_\alpha t^\alpha/\alpha!$.

The expressions (12) and (13) are our final results written in a condensed form. The main values of the Minkowski functionals are inferred from (12) or (13) by comparing the coefficients of $t^\alpha/\alpha!$, $\alpha = 0,..., d$, on both sides.

With the expansion $1 - e^{-\mu} = \sum_{\alpha=0}^{d} \Phi_\alpha t^\alpha/\alpha! + O(t^{d+1})$, we obtain from (13),

$$\bar{M}_\alpha(D) = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} M_\beta(D)\, \Phi_{\alpha-\beta}, \qquad \alpha = 0,..., d \tag{14}$$

where, for example,

$$\begin{aligned} \Phi_0 &= 1 - e^{-\rho m_0}, \qquad & \Phi_2 &= (\rho m_2 - m_1^2 \rho^2)\, e^{-\rho m_0} \\ \Phi_1 &= \rho m_1 e^{-\rho m_0}, \qquad & \Phi_3 &= (\rho m_3 - 3 m_1 m_2 \rho^2 + m_1^3 \rho^3)\, e^{-\rho m_0} \end{aligned} \tag{15}$$

We note that the quantities m_α and $M_\alpha(D)$ still depend on the dimension d; see Table I. The contributions to $\bar{M}_\alpha(D)$ from $M_\alpha(D)$ in (14) describe effects due to the finite size and shape of the observation window D. If D is dilatated uniformly by the factor λ, then $M_\alpha(\lambda D) = \lambda^{d-\alpha} M_\alpha(D)$. Therefore, the relations (14) yield

$$\Phi_\alpha = \lim_{\lambda \to \infty} \frac{\bar{M}_\alpha(\lambda D)}{M_0(\lambda D)}, \qquad \alpha = 1,..., d \tag{16}$$

In other words, Φ_α denotes mean values of the Minkowski functionals [cf. Eq. (4)] per unit volume for the grain model.

For $d = 1$, 2, 3, the explicit expressions of the mean Euler characteristics per particle $\bar{\chi}_d = \omega_d \Phi_d(d)/\rho$ read[9]

$$\bar{\chi}_1 = e^{-n}, \qquad \bar{\chi}_2 = \left(1 - \frac{u^2}{4\pi f} n\right) e^{-n}, \qquad \bar{\chi}_3 = \left(1 - \frac{hs}{4\pi v} n + \frac{\pi s^3}{384 v^2} n^2\right) e^{-n} \tag{17}$$

with $n = \rho m_0(d)$. In the special case of monodisperse balls, $\bar{\chi}_1$ and $\bar{\chi}_2$ agree with Okun's result.[1] However, his recurrence formula [$\bar{\chi}_d = \partial(n\bar{\chi}_{d-1})/\partial n$, in the present notation] yields $\bar{\chi}_3 = (1 - 3n + n^2)\, e^{-n}$, in disagreement with the exact $\bar{\chi}_3 = (1 - 3n + 3\pi^2 n^2/32)\, e^{-n}$. The Euler characteristic is related with the integral Gaussian curvature via the combinatorial analogue of the Gauss–Bonnet theorem. The term $3\pi^2 n^2/32$ in $\bar{\chi}_3$ arises from the singular Gaussian curvature at the points where three spheres intersect. It can be shown[(11)] that Okun's recurrence relation is valid in the case of oriented, parallel hypercubes, where the group of motions is restricted to translations.

The mean Euler characteristics of lattice sets are polynomials in $p = 1 - e^{-n}$, which coincide with the matching polynomials known from percolation theory.[(12)] Moreover, $\bar{\chi}_2(p)$ vanishes at the exact threshold value p_c for percolation on a two-dimensional self-matching lattice. In the case of continuum percolation, exact values of p_c are not known, but some efforts have been made,[(13–15)] based on numerical estimates and excluded-volume arguments, to infer empirical bounds on p_c. Motivated by these attempts to arrive at practically useful percolation criteria, we looked for a possible connection between n_c of continuum percolation and the zeros n_0^d of $\bar{\chi}_d(n)$. In Table II, numerical n_c data for some grain shapes are compared with the zeros

$$\begin{aligned} n_0^{(2)} &= \rho_0 f = \frac{4\pi f}{u^2} \\ n_0^{(3)} &= \rho_0 v = \frac{48hv}{\pi^2 s^2}\left[1 - \left(1 - \frac{\pi^3 s}{6h^2}\right)^{1/2}\right] \end{aligned} \tag{18}$$

obtained from Eqs. (17); $n_0^{(3)}$ denotes the smaller one of the two zeros of $\bar{\chi}_3(n)$.

The data tempted us to speculate that $n_0^{(2)} \leqslant n_c^{(2)}$ and $n_0^{(3)} \geqslant n_c^{(3)}$ might possibly hold as universal bounds for n_c. As a further support for this

Table II. Threshold Values for the Percolation Density Parameter and Zeros of the Euler Characteristic[a]

$d=2$ discs	$d=2$ sticks	$d=3$ balls	$d=3$ discs
$n_c = 1.12$[b]	$l^2 \rho_c = 5.7$[b]	$n_c = 0.34$[b]	$\rho_c = 0.19$[c]
$n_0 = 1$	$l^2 \rho_0 = \pi$	$n_0 = 0.38$	$\rho_0 = 0.22$

[a] The sticks in $d=2$ are rectangles of length l and vanishing breadth. The discs in $d=3$ are cylinders of radius $r = 0.5$ and vanishing height.

[b] Simulation data from ref. 14.

[c] Simulation data from ref. 16.

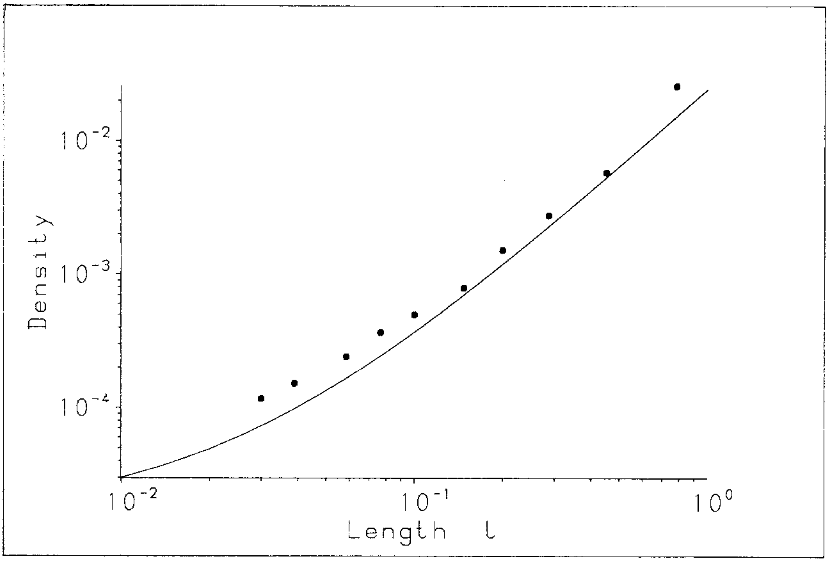

Fig. 1. Threshold densities[15] for percolating cylinders (length l, radius $r = 0.015$) in $\mathbb{R}^3$ compared with $\rho_0(l)$ (full curve).

conjecture, we plotted in Fig. 1 the $\rho_c(l)$ values for randomly located and orientated cylinders of length l and fixed radius r in comparison with $\rho_0(l)$, which tends toward $2/(\pi r l^2)$ for $l/r \to \infty$.

It would be interesting to check whether the second zero of $\bar{\chi}_3(n)$ concurs with void percolation, but we are not aware of simulation data for this case.

Finally, we remark that Eq. (12) with a window domain $|D| = L^d$ leads to a grain-shape-dependent finite-size correction in the Euler characteristics such that $n_0^{(d)}(L)$ increases $\propto 1/L$ with decreasing L. A similar behavior of $n_c^{(3)}(L)$ was observed[16] for percolating discs in $\mathbb{R}^3$.

REFERENCES

1. B. L. Okun, *J. Stat. Phys.* **59**:523 (1990).
2. S. B. Gray, *IEEE Trans. Computers* **20**:551 (1971).
3. T. Hofsäss and H. Kleinert, *J. Chem. Phys.* **86**:3565 (1987).
4. D. M. Anderson, H. T. Davis, and L. E. Scriven, *J. Chem. Phys.* **91**:3246 (1989).
5. A. L. Melott, *Phys. Rep.* **193**:1 (1990).
6. H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie* (Springer, 1957).
7. P. Hall, *Introduction to the Theory of Coverage Processes* (Wiley, 1988).
8. L. A. Santaló, *Integral Geometry and Geometric Probability* (Addison-Wesley, 1976).
9. P. Davy, *J. Appl. Prob.* **13**:714 (1976).

10. H. G. Kellerer, *Z. Wahrsch. Verw. Geb.* **67**:63 (1984).
11. K. R. Mecke and H. Wagner, to be published.
12. M. F. Sykes and J. W. Essam, *J. Math. Phys.* **5**:1117 (1964).
13. H. Scher and R. Zallen, *J. Chem. Phys.* **53**:3759 (1970).
14. G. E. Pike and C. H. Seager, *Phys. Rev. B* **10**:1421 (1974).
15. I. Balberg, *Phil. Mag. B* **56**:991 (1987).
16. E. Charlaix, *J. Phys. A: Math. Gen.* **19**:L533 (1986).

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